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THE KELVIN-HELMHOLTZ INSTABILITY WITH
ARBITRARILY ORIENTED BODY FORCE

by

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ABSTRACT

This thesis is a study of the interfacial stability of two parallel superposed inviscid streams of fluid. The problem is a generalization of the Kelvin-Helmholtz model, with the new feature being the addition of a component of the body force tangential to the flow direction. The results are applied to the stability of the molten liquid-gas interface on an ablating re-entry body.

Using a linearized normal mode approach for the stability analysis, a dispersion equation relating the complex phase velocity to the wave number of the assumed disturbances is found. Explicit dependence on the density ratio, relative interface velocity, surface tension, and the body force components is expressed in this result.

The dispersion relation is first studied for the case with a zero tangential body force component in order to gain insight into other mechanisms affecting the stability of the system. For the complete problem, the tangential component is found to be always destabilizing. The perpendicular component is destabilizing or stabilizing depending on whether it is directed away from or toward the heavier medium. The relative interface velocity introduces a mechanism which acts to destabilize the flow independently of other effects.

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LIST OF SYMBOLS

| | |
|---------------|---------------------------------------------------------------------|
| c | complex phase velocity |
| d | thickness of the liquid layer |
| D | differential operator d/dy |
| $f(y)$ | complex disturbance amplitude function for u' |
| $F(k, \beta)$ | function defined by (5-7) |
| F_x, F_y | Froude numbers corresponding to x, y components of the body force |
| g_x, g_y | x, y components of the body force per unit mass |
| $h(y)$ | complex disturbance amplitude function for v' |
| i | $(-1)^{1/2}$ |
| k | wave number of the disturbances |
| p, P | fluid pressure |
| p_1, p_2 | roots of the quadratic (3-27) |
| $q(y)$ | general complex disturbance amplitude function |
| $r(y)$ | complex disturbance amplitude function for ρ' |
| $s(y)$ | complex disturbance amplitude function for η |
| t, T | time |
| τ | surface tension coefficient |
| u, U | x -component of velocity |
| v, V | y -component of velocity |
| W | Weber number |

| | |
|-----------|--------------------------------------------------------------------|
| x, X | coordinate along wall (i.e., flow direction) |
| y, Y | coordinate normal to wall, measured from the undisturbed interface |
| β | exponential density factor, defined by (3-25) |
| Δ | interface jump quantity, defined by (3-22) |
| η | interface disturbance function |
| θ | angle measured on body, defined in Fig. 1. |
| λ | wavelength of the disturbances |
| $\pi(y)$ | complex disturbance amplitude function for p' |
| ρ | fluid density |
| ω | complex frequency of the disturbances |

Subscripts:

| | |
|-----|-----------------------------------|
| i | imaginary part |
| r | real part |
| L | lower fluid (liquid) |
| u | upper fluid (gas) |
| R | reference quantity |
| $*$ | dimensional quantity |
| $'$ | disturbance quantity |
| o | basic flow quantity (dimensional) |

~ for cutoff wave number defined by (4-5a)

- for cutoff wave number defined by (5-9c)

Note: Capital letters for dependent and independent variables
denote dimensional quantities; lower case letters denote
dimensionless quantities.

ERRATA

| <u>Page</u> | <u>Line</u> | <u>Correction</u> |
|-------------|------------------|----------------------------------------|
| 12 | 6 | "analysis" to "analyses" |
| 24 | equation (2-2) | "function" to "function of" |
| 25 | 1 | Insert "either" after "implies" |
| 31 | equations (3-6) | Replace "h" with "d" |
| 36 | equations (3-16) | "s(y)" to "s" (a constant) |
| 36 | 7 | Number the equation (3-15) |
| 43 | 4 | " $c \neq 1$ " to " $\neq -1$ " |
| 44 | 3 | Add parentheses at end of line. |
| 67 | 17 | "F _y " to "F _y " |

CHAPTER I

INTRODUCTION

A. Motivation and Previous Work

In recent years the problem of providing heat protection for re-entry vehicles has become of great importance. This has led to various investigations of ablation processes and the related flow and heat transfer features. The study of the ablation of skin materials which melt before vaporizing has been a frequently examined problem. Various physical mechanisms such as high heat-transfer rates, large deceleration forces, and ablative liquid detachment are involved in such ablation problems. In studying the effects of these mechanisms it becomes important to analyze the stability of the molten ablative liquid-gas interface which is subject to the extreme conditions encountered on re-entry into the Earth's atmosphere. It is this interfacial stability problem that will be of concern in the present paper.

Most analyses which have dealt with such molten layers have been limited to the stagnation region of the re-entry body. Several reasons may be pointed out for this restriction. In the stagnation region, the most severe heat transfer processes have been shown to occur, [1]. Also, the deceleration force acts

perpendicularly with a destabilizing orientation. It appears that only Cheng [2] has dealt with the stability problem away from the stagnation point, where a component of the deceleration force parallel to the interface plays a role.

Associated with these two possible regions of interest are two classically studied interfacial instability problems:

- 1) the Rayleigh-Taylor instability,
- 2) the Kelvin-Helmholtz instability.

The Rayleigh-Taylor problem generally refers to the stability of two superposed viscous incompressible fluids, initially at rest, subject to both surface tension and to a single body force component normal to the interface. It has been shown by various investigators (Taylor [3], Bellman and Pennington [4], Reid [5]) that for unbounded fluids the configuration is stable so long as the body force is directed toward the denser medium. If the body force direction is reversed, the system is unstable, regardless of the viscous effects. The surface tension serves to introduce a cutoff wave number above which all disturbances are damped out. Viscosity serves only the role of lowering the amplification rates of the disturbances, but it can never completely stabilize this

situation. The cutoff wave number is independent of the role of viscosity depending directly on the magnitude of the body force and the density difference of the fluids and inversely on the surface tension coefficient. (See equation (4-4).)

Thus instabilities of the type excited primarily by the body force normal to the two fluid interface, as described here, are called Rayleigh-Taylor instabilities. It is this mechanism which is the dominant destabilizing influence in the stagnation region of a re-entry body. (See Figure 1.)

The Kelvin-Helmholtz problem generally refers to the stability of two superposed incompressible inviscid parallel flows, subject to surface tension at the interface and to the normal component of the body force alone. The effect of this body force on the system is the same as in the Rayleigh-Taylor case, the new feature here being the relative velocity of the two streams. This is the physical mechanism usually attributed to the creation of ocean waves by wind, although Miles [6] has recently shown this claim to be invalid.

In describing the Kelvin-Helmholtz instability mechanism, which is absent of any viscous effects, the interface now

represents a discontinuity surface for the velocities of the two fluids, and, as such, may be interpreted as a vortex sheet. Using an energy balance equation across the interface, Cheng [7] has shown that the interfacial vortex region may be visualized as a thin viscous region within which there is a large velocity gradient. This large gradient gives rise to the Reynolds stress excitation necessary for such interfacial instabilities. Thus the vortex sheet acts to create a seemingly artificial energy source which is a representation of the physical process of viscous excitation through the Reynolds stress. (Such replacement of a viscous shear layer by a vortex sheet is not uncommon in fluid mechanics. Recall that a great deal of three-dimensional airfoil theory, such as Prandtl's lifting-line theory, depends initially on this very representation.)

Thus interfacial instabilities of superposed fluids excited primarily by the viscous action through Reynolds stresses are termed Helmholtz instabilities (See Cheng [7]). It should be added at this point that in the study of the stability of viscous boundary layer flows, the term Tollmien-Schlichting instability is associated with the instability occurring as a result of amplification by viscosity of infinitesimal disturbances within the

layer. Thus, both the Helmholtz and the Tollmien-Schlichting instabilities are created by the transfer of energy from the main flow to the disturbance flow through the action of viscous stresses. However, in considering the instabilities which may occur in a system of viscous superposed flows, these two mechanisms may play independent roles. For example, a Tollmien-Schlichting instability may arise within a viscous layer, and consequently lead to an instability of the interface. Such a break-up of the interface would not be due to the Helmholtz mechanism, but rather to a transfer of energy as a result of the Tollmien-Schlichting instability. To the author's knowledge, no stability analysis related to viscous superposed flows has explicitly distinguished these two instability mechanisms and their relative importance. The aim of such a distinction would be to determine which instability is most likely to occur first, and hence which is the dominant mechanism in the flow break-up.

Ostrach and Koestel [8] have pointed out that experiments indicate that an interfacial instability will usually precede a Tollmien-Schlichting instability within a fluid layer. It has been shown that the interfacial instability may even occur if the fluid layers are both in laminar motion. If such a result could be

verified in general, then examination of the instability of the interface could be made independently of knowledge of the stability of the fluid layers. Although such verification has not yet been achieved, many analyses (e.g., [9], [10], [11], [12]) do assume the instability of the interface to be the dominant mechanism in destroying the given flow pattern.

Therefore, if the Kelvin-Helmholtz instability is now considered as a special case of the Helmholtz instability in the limit of zero viscosity, then one may state that the Kelvin-Helmholtz model is a representation of a case when the interfacial instability precedes any instability in the fluid layers. Since then viscosity within the layers would no longer be the dominant destabilizing influence, one may assume the fluids to be inviscid, and, as discussed, represent the viscous shear layer separating the flows by a vortex sheet. This argument appears to be a proper justification for the inviscid Kelvin-Helmholtz model. Of course, it should be realized that the stability analysis ensuing from this model will yield somewhat pessimistic results. This latter fact is due to the omission of the possible stabilizing influence of viscosity at the interface (see Cheng [2]), and for short waves (Miles [16]).

Thus the Kelvin-Helmholtz model may be characterized by the fixed relative velocity (at the interface) of the superposed streams. It will be shown that it is the magnitude of this relative velocity which indicates the "strength" of the Kelvin-Helmholtz instability. The effect of the body force normal to the interface is to be interpreted as a superposition of the Rayleigh-Taylor mechanism upon the given parallel streams.

From Figure 1, it would now appear that away from the stagnation point the Kelvin-Helmholtz configuration may be a proper model locally on a re-entry body, subject to the neglect of viscosity. There is, however, in the present case of melting ablation an additional factor, namely the body force g_x parallel to the interface. The magnitude of this component, as well as the normal component g_y will vary from point to point because of the body curvature. Locally, the flow picture may be interpreted as in Figure 2. This will be used later to describe the assumed model for the analysis.

Before proceeding with discussion of the stability problem relevant to the ablation study, further mention of earlier work on the Kelvin-Helmholtz problem is of value. As discussed above, the Kelvin-Helmholtz instability is characterized by the

relative velocity of the two fluids as a result of the assumed jump discontinuity in velocity at the interface. Goldstein [11] and Taylor [12] both tried to improve this model by eliminating the jump changes in both velocity and density at the interface. This was done, still assuming inviscid fluids, by inserting transition layers between the main flows of interest. These additional layers allow a means for the velocity and density to change continuously from one main stream to the other. In this way, Goldstein and Taylor were able to show that the jump discontinuities in the original model accentuated the instability. Again, one might interpret these transition layers as a rough attempt to represent the viscous shear layer which, in reality, joins the two main flows. A similar approach was used by Drazin and Howard [14] for a fluid with a continuously stratified density distribution.

Perhaps the most active investigator of these problems is John W. Miles, who, in several series of articles (see reference [15]), has studied the various Kelvin-Helmholtz and related problems, and their ranges of applicability. In addition to his previously mentioned work [6] , Miles [16] has shown that viscous effects may be significant, especially for short waves. He [6] also has generalized the Kelvin-Helmholtz model for

parallel shear flows by considering the heavier fluid as a viscous medium. The body force is oriented so that the system is stable in the Rayleigh-Taylor sense. The principal application of the results is to the flow of a light inviscid fluid over a viscous liquid, with an air-oil interface typical of a calculation agreeing well with observed results. This model also leads to the conclusion that this Kelvin-Helmholtz stability is not the proper mechanism for the generation of water waves at commonly observed wind speeds. Namely, Miles finds that his generalized Kelvin-Helmholtz model is only good for a heavy liquid of relatively large viscosity. It would thus appear that such a model would be applicable to the stability of an ablative interface near the stagnation point, mainly because of the large viscosities of the liquid layers involved.

With regard now to the stability problem associated with the ablation configuration away from the stagnation point, it can be noted that most related studies have neglected the component of the body force g_x parallel to the interface. The analyses thus become various generalizations of the previously defined Kelvin-Helmholtz problem. The work of Cheng [2] appears to be the only analysis which accounts for both body force components.

Feldman [17] , for example, states that this parallel component of the body force serves only to change the form of the liquid film velocity profile. This implies that only the instabilities in planes normal to the flow direction are of consequence. Thus, in reference [17] Feldman finds the behavior of such equivalent Rayleigh-Taylor instabilities. (It can be shown that if all gradients in the flow direction vanish, the stability analysis in planes normal to the flow is independent of the flow velocity. Hence, this analysis corresponds to a Rayleigh-Taylor problem.) It will be, however, one of the purposes of this paper to show explicitly the effect of the tangential component of the body force on the gas-liquid interface instability. The results will then be compared with those of Cheng [2] . The results will show Feldman's contention regarding this body force component to be incorrect.

In reference [17] , Feldman does point out the two primary mechanisms which can lead to a loss of liquid through instability of the gas-liquid interface. One is that at sufficiently high liquid Reynolds numbers, energy can be transferred by viscous effects from the gas stream into the liquid layer at such a rate that liquid entrainment by the gas stream may result. The

other mechanism is provided by the Rayleigh-Taylor instability dominant in the stagnation region of the body. Neither of these influences, however, account for the effect of the tangential component of the body force downstream of the stagnation point. The effect of this component has, however, been shown to play an important role in the flow characteristics. For example, Ostrach, Goldstein, and Hamman [18], in studying flow and heat transfer characteristics of melting ablation layers, have pointed out that in determining conditions away from the stagnation point, it is important to include all effects of the deceleration force. Since this force will actually oppose the downstream flow of liquid (i.e., flow away from the stagnation point), new features may arise. As a case in point, they have shown the conditions under which liquid can be forced upstream and eventually accumulate at some position away from the nose of the body. Thus the predominant role of the tangential component of the body force is illustrated.

Further, Cheng [2], using a stability analysis involving an approximate energy integral approach, has found that this tangential body force is always destabilizing in the nose region (i.e., in the region adjacent to the stagnation point), and that in

the transonic region of a blunt body, this component may cause an instability (in a generalized Rayleigh-Taylor sense) more serious than that existing in the stagnation region. Such results certainly differ from Feldman's contentions, thereby making further investigation of this situation worthwhile.

Several other stability analysis related to the ablation process may be discussed. Cheng [7] also studied the Rayleigh-Taylor problem with the same approach as he used in [2], achieving approximate results for the maximum amplification rate of small disturbances, as well as rederiving the previously known aspects of the effects of surface tension and viscosity. Feldman in [18] actually applied, to the problem of an ablating heat shield, the results of an earlier stability analysis [19], in which a linear velocity profile was assumed in each fluid. Miles [16] corrected some mathematical errors in Feldman's work [19] that should allow the analysis to be applicable in the limit as the gas-liquid density ratio tends to zero. Miles, however, notes that these results still do not agree with observation.

Another related analysis by Chang and Russell [9] serves as a generalization of the Kelvin-Helmholtz problem for

parallel streams passing over an infinite flat plate. The gas is treated as an inviscid compressible medium, with the liquid as a viscous incompressible fluid initially at rest. Using a linearized compressible flow theory, the subsonic and supersonic cases, together with inviscid and viscid limits, are examined. This analysis leads to quantitative criteria governing the stability of their assumed configurations. The results appear valid for all cases considered, but application to the ablation problem may be limited to the nose region of very blunt bodies where the tangential component of the body force may be neglected.

A similar analysis to the previous one was performed by Plesset and Hsieh [10] in which compressibility is accounted for in both fluids with, however, both viscosity and surface tension neglected. The results obtained are valid generalizations of known Kelvin-Helmholtz results accounting for the compressibility effects. Again, only a body force normal to the interface was assumed.

Another series of papers, although not directly applicable to the ablation problem, is of interest as regards the problem of viscous superposed flows. This is the work of Benjamin [20], Yih [21], and Kao [13]. They have examined the stability of

viscous flows down inclined planes, assuming the flow is always parallel to the incline. In the respect that such a flow would involve body force components both parallel and perpendicular to the flow direction, it is not unlike the stability problem for a molten ablation layer away from the stagnation region (see Figure 1). An important difference is that in the first case the one body force component is parallel to the flow serving as its driving mechanism, whereas in the ablation problem it is anti-parallel to the flow and hence a retarding mechanism. In this latter case, the pressure gradient must drive the flow in a direction opposite to that of the body force component tangent to the surface of the body.

Kao [13] took Yih's work, which applied to a single fluid only, and extended it by allowing another fluid to be superposed on the first, with a free surface bounding this upper fluid from above. This configuration is closer to the two-fluid stability problem of interest herein, except that in the ablation situation there is no free surface bounding the two flows. Kao analyzed his results only for the limiting cases of small and large wave numbers. Among other results he found, assuming a fully-developed parabolic velocity distribution in each fluid, that even

for the case in which the upper fluid density is greater than that of the lower fluid, there still exists a minimum critical Reynolds number below which all disturbances are damped. This result demonstrates a stabilizing influence of viscosity which can be enough to stabilize a configuration which is subject to the destabilizing effect of the Rayleigh-Taylor mechanism. As mentioned earlier, however, in the usual Rayleigh-Taylor problem, viscosity cannot serve to stabilize the flow completely, but only to lower the amplification rates. Thus Kao's results add a significant feature which may be applicable to the stability problem associated with the ablation process. This same feature has been also demonstrated by Cheng [2] . From the earlier discussion, recall that such a stabilizing influence of viscosity will be absent in the Kelvin-Helmholtz model.

B. Objectives of the Present Work

The above survey of relevant analyses now provides a basis for the present study of the effect of a body force component opposed to the flow direction on the Kelvin-Helmholtz instability. This will be done with a somewhat simplified model from that which physically exists in the ablation problem.

The analysis will be kept two-dimensional, noting that it has been shown generally that for the Kelvin-Helmholtz flows the most destabilizing disturbances are those in the flow direction [22] . Hence, neglect of fully three-dimensional disturbances will yield pessimistic results from the analysis.

Further, viscosity will be neglected throughout. This assumption, as has been discussed, is the most serious limitation of the theory to be presented. In light of the previous mention of Miles' conclusion [16] regarding the effect of viscosity on short waves, and Kao's result [13] on the stabilizing influence of viscosity, the assumption of inviscid flows may also lead to somewhat pessimistic results.

Curvature effects of the body will be neglected by assuming the bounding wall for the liquid is a flat plate of infinite extent. (See Figure 2.) In this way all gradients in the direction of the basic flow (i.e., parallel to the wall) may be taken to zero. Physically, such a simplification is especially justified in the liquid layer since it is so thin that gradients normal to the wall would be expected to dominate those parallel to the wall.

Thus the configuration to be studied is a generalization

of the Kelvin-Helmholtz model. The main feature of the generalization is the introduction of the tangential body force component opposed to the motion of the fluid. The stability analysis will be done using the linearized normal mode approach. This will yield from the linearized disturbance equations two second-order homogeneous ordinary differential equations subject to homogeneous boundary conditions and to an interface condition. The resulting characteristic equation relates the complex phase velocity (or the complex angular frequency) to the wave number of the same. The various physical mechanisms are represented by dimensionless parameters which first appear through the governing equations and boundary conditions. Such a characteristic equation is generally called the dispersion relation, and, from it, will be derived the stability features of the assumed configuration. Namely, certain criteria for stability may be found as a function of the dimensionless parameters of the problem, thereby indicating the relative stabilizing and destabilizing influences.

CHAPTER II

THE BASIC FLOW

A. Introduction and Assumptions

The basic flow pattern whose stability is now to be studied is the steady incompressible two-dimensional parallel flow of two superposed inviscid fluids over an infinite flat plate, subject to a gravitational field with force components parallel and perpendicular to the flow direction. See Figures 1 and 2.

Again, the main goal of the present study is to find the influence on the interfacial stability of the body force component g_x parallel to the gas-liquid interface. In formulating a generalized model to study this effect, several assumptions have already been discussed (e.g., two-dimensionality, neglect of viscosity and body curvature). These and the remaining assumptions are examined in the following, with proper justification offered for each. Then, the conclusions to be drawn will be shown to demonstrate some of the salient features of this stability problem.

For completeness, all of the basic assumptions to be made in accordance with Figure 2 are now listed, with the

important aspects discussed in the following:

1. Both flows are assumed initially steady and parallel to a plane wall at $Y = -h$.
2. Each fluid is inviscid.
3. The density of the upper fluid is assumed constant. The lower fluid is a stratified medium such that constant density surfaces are parallel to the boundary wall. It will also be considered "incompressible" in the sense that the density of any given fluid particle remains constant as its motion is followed. Thus

$$\frac{D\rho^0}{Dt} = U \frac{\partial \rho^0}{\partial X} + V \frac{\partial \rho^0}{\partial Y} = 0,$$

thereby coupling the velocity components to the density variation.

4. The two-fluid configuration is statically unstable in the Rayleigh-Taylor sense. Explicitly, $\rho_u < \rho_L$ at the interface $Y = 0$, with the body force normal to the interface directed away from the denser fluid.

5. The direction of the tangential body force component is opposed to the direction of flow. With reference to the ablation problem such orientation usually corresponds to flow away from the stagnation point (for discussion of this point, see Ostrach, Goldstein, and Hamman [1]).

6. Surface tension is included at the interface.

7. The gas-liquid density ratio (ρ_u/ρ_L) and the velocity ratio (U_L/U_u) are much smaller than unity. From ablation data [1], $\rho_u/\rho_L = O(10^{-5})$ and $U_L/U_u = O(10^{-3})$.

The first assumption is made for convenience of the analysis. Since the present work is a generalization of the Kelvin-Helmholtz problem, the assumption of a steady basic flow here is consistent with this original model.

Although, as seen from Figure 1, the body force components vary from point to point on the body surface as a function of the angle θ , the assumption that the surface is a plane wall is equivalent to saying that only a small segment of the body

surface is to be considered. Thus the problem is one for which a locally parallel flow may be assumed, as indicated by Figure 2.

The assumption that both fluids are inviscid is, perhaps, the most serious limitation of the ensuing analysis, but again is consistent with the Kelvin-Helmholtz interfacial model. With regard to the upper fluid, this assumption is quite reasonable, since it is expected that at the interface the pressure and inertia forces would greatly dominate frictional effects due to a viscous shear layer. This is in agreement with Chang and Russell in reference [9]. The neglect of viscosity in the liquid layer is, however, not justified so easily in light of evidence of the actual "slow viscous flow" behavior of ablating materials (e.g., see Ostrach, Goldstein, Hamman [1], [18]). Here the results will be considered in the light of justification of the inviscid Kelvin-Helmholtz model, as discussed in Chapter I.

Although the assumption of inviscid flows will yield some proper aspects of the interfacial instability, the stabilizing influence of viscosity as found, for example, by Cheng [2] should be kept in mind. Generally, it might be noted that even though viscosity may not completely stabilize an interfacial instability as in Cheng's case, it generally will at least lower the amplification

rate of the disturbances (as for the Rayleigh-Taylor problem). It is these above-named stabilizing influences of viscosity on the interfacial instability of the present problem that will be lacking in the final results. For a complete understanding of the effect of viscosity on the melting ablation problem, a more complete analysis than that presented here is needed. Perhaps then one could ascertain, as discussed in the Introduction, the complete roles of both the Helmholtz instability of the interface and the Tollmien-Schlichting instability within the fluid layers.

The assumption 3) is made to allow the main effect of the upper fluid to be one of an inertial nature only. The effect of compressibility in the upper fluid is discussed by Chang and Russell [9]. In allowing for stratification of the lower fluid, the liquid density variation that would result in the ablation problem from aerodynamic heating is being taken into account. (It will be seen later that this assumption is also necessary for the inclusion of the body force component parallel to the flow direction.)

In making assumptions 4) and 5), it is noted again that a "static" instability mechanism of the Rayleigh-Taylor type adds to a "dynamic" instability of the Kelvin-Helmholtz type. Such "static" influence is indeed the case in the ablation problem of

interest, where the body force components are oriented in the manner indicated in assumptions 4) and 5).

The inclusion of surface tension according to assumption 6) is in accord with the physics of a problem of interfacial stability. It would be anticipated that surface tension plays a stabilizing role, and this will be shown to be the fact in the present case.

B. The Governing Equations

With these assumptions now delineated, the equations governing the basic flow may be written. The geometry indicated in Figure 2 is used, denoting the lower fluid by the subscript L and the upper fluid with the subscript u, and setting our coordinate system so that $Y = 0$ corresponds to the undisturbed interface and $Y = -d$ corresponds to a fixed rigid wall. Since from assumption 1) the flow is unbounded in the X-direction, the line $X = 0$ is not unique, that is, its choice is arbitrary. Because of this fact one would expect the velocity to be fully-developed, that is, independent of X. The pertinent continuity and momentum equations for each fluid then read

$$0 = \frac{\partial V}{\partial Y} \quad (2-1a)$$

$$0 = - \frac{\partial P}{\partial X} - \rho^0 g_X \quad (2-1b)$$

$$0 = - \frac{\partial P}{\partial Y} + \rho^0 g_Y \quad (2-1c)$$

In (2-1b) and (2-1c), the consequence of equation (2-1a) has been applied, namely, $V = 0$ in each region. As a result of equations (2-1), the velocity distribution, in each fluid, is given by

$$U = \text{arbitrary function } Y; \quad V \equiv 0. \quad (2-2)$$

Note U is an arbitrary function of Y because in an inviscid fluid the X -component of velocity is not required to satisfy any boundary conditions in Y (namely at the wall, for Y infinite, and at the two-fluid interface).

From (2-1b) and (2-1c), eliminating P by cross-differentiation

$$g_X \frac{\partial \rho^0}{\partial Y} = - g_Y \frac{\partial \rho^0}{\partial X} ,$$

which implies that

$$\rho^0 = \text{fcn} (-Y g_Y + X g_X), \text{ or } \rho^0 = \text{constant. (2-3)}$$

Similarly,

$$P = \text{fcn} (-Y g_Y + X g_X). \quad (2-4)$$

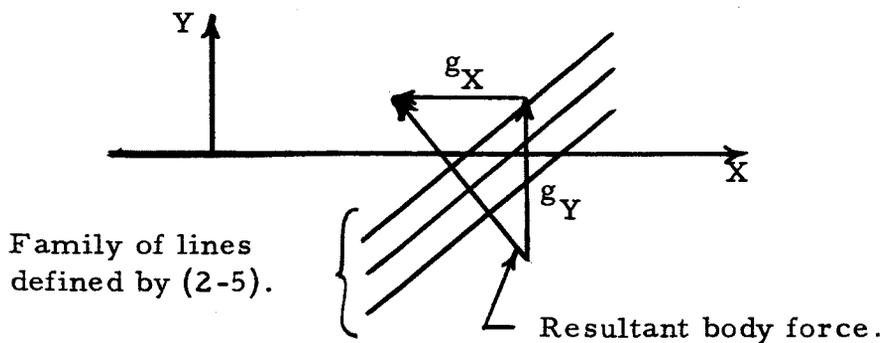
Thus, from (2-3) or (2-4), constant density (pressure) planes are defined by the lines

$$- Y g_Y + X g_X = \text{constant} \equiv -K$$

or

$$Y = K + \left(\frac{g_X}{g_Y} \right) X \quad (2-5)$$

This is the family of straight lines normal to the resultant body force whose slope is given by (g_Y/g_X) ; schematically,



Thus it is found that the flow is stratified along lines defined by equation (2-5), and not along constant Y-lines according to assumption 3). (An exception is the case $\rho^0 = \text{constant}$, hence $P = \text{constant} - \rho^0 (-Y g_Y + X g_X)$. Here again the pressure is constant only along the lines defined by equation (2-5), although the density is everywhere constant.)

Physically, the consequence of fluid stratification along lines defined by equation (2-5) is that the interface must also be defined by a member of that family. This is seen easily since in the inviscid case the pressure must be continuous across the interface. Thus, according to the governing equations (2-1) the interface also must be described by a member of the family of straight lines (2-5).

Thus, there is an apparent contradiction. Namely, that the flow cannot be parallel to the wall at $Y = -d$, but only parallel to the family of lines defined by equation (2-5). This latter statement implies a non-zero Y-component of the velocity, contrary to the result (2-2). In other words, according to the assumptions, the basic flow is always parallel to the bounding wall, whereas the governing equations (2-1) imply that the flow is always parallel to the direction defined by (2-5).

In order to avoid this discrepancy, it is now further assumed that within the flow regions of interest the pressure gradient in the x-direction can be neglected. This assumption will then limit the analysis to regions wherein such a limitation is valid locally. Also it will be seen that this assumption will restrict the allowable class of disturbances that may be considered within the stability analysis.

Analytically, the previous statements imply that the governing equations (2-1) yield (together with (2-2))

$$U = U(Y), \quad V = 0.$$

$$\frac{\partial P}{\partial X} = 0. \tag{2-6}$$

$$\frac{dP}{dY} = \rho^0(Y) g_Y.$$

This set now defines the basic flow whose stability will be studied in the following chapters.

CHAPTER III

FORMULATION OF THE STABILITY PROBLEM

A. The Disturbance Equations

The method of stability analysis to be applied is that usually termed the normal mode approach. This is a linear theory arrived at by linearizing the complete equations for infinitesimal disturbances about the basic flow. Thus the solution sought is the behavior of the system relative to infinitesimal disturbances of a particular nature to be assumed.

In mathematical language, a time-independent basic flow is known in terms of $U(Y)$, $\rho^0(Y)$, and $P(Y)$. The stability problem is then an initial value problem whose dependent variables are slightly (infinitesimally) different from the time-independent solution. As time increases without bound, if the solution approaches the basic flow solution, the initial flow is said to be stable; otherwise, it is unstable. Actually, also acceptable is a sinusoidal motion (non-increasing or non-decreasing with time) superposed over the basic motion as a (marginally) stable motion. In many stability analyses it is this marginal state that is sought, since it is considered the boundary between stable and unstable motions. Instability here does not imply turbulent

motion ensuing, but rather some immediate departure from that which has been termed the basic flow.

The full equations pertinent to the flow situation are (valid in each fluid):

$$\text{Continuity: } \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad (3-1)$$

"Incompressibility":

$$\frac{\partial \rho^{\circ}}{\partial T} + U \frac{\partial \rho^{\circ}}{\partial X} + V \frac{\partial \rho^{\circ}}{\partial Y} = 0 \quad (3-2)$$

Momentum:

$$\rho^{\circ} \frac{\partial U}{\partial T} + \rho^{\circ} U \frac{\partial U}{\partial X} + \rho^{\circ} V \frac{\partial U}{\partial Y} = - \frac{\partial P}{\partial X} - \rho^{\circ} g_X \quad (3-3)$$

$$\rho^{\circ} \frac{\partial V}{\partial T} + \rho^{\circ} U \frac{\partial V}{\partial X} + \rho^{\circ} V \frac{\partial V}{\partial Y} = - \frac{\partial P}{\partial Y} + \rho^{\circ} g_Y$$

For convenience in later determining the important physical parameters of the problem, it is now practical to put the equations above in non-dimensional form. The following reference quantities are thus defined:

$$U_R = U_L \Big|_{y=0} \equiv U_{0L}$$

$$\rho^o_R = \rho_L \Big|_{y=0} \equiv \rho_{0L}$$

$$L_R = d \quad (3-4)$$

$$P_R = \rho_{0L} U_{0L}^2$$

$$T_R = d/U_{0L}$$

Note the reference quantities for the density and velocity are taken with respect to the liquid layer since it will be for this fluid that the main characteristics of the stability problem will be seen.

In dimensionless form, using (3-4), the equations (3-1, 2, 3) become

$$a) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$b) \quad \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 \quad (3-5)$$

$$\begin{aligned} \text{c) } \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] &= - \frac{\partial p}{\partial x} - \frac{\rho}{F_x} \\ \text{d) } \rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] &= - \frac{\partial p}{\partial y} + \frac{\rho}{F_y} \end{aligned} \tag{3-5}$$

In (3-5c) and (3-5d) F_x and F_y are the Froude numbers defined corresponding to g_x and g_y , respectively; thus,

$$F_x = \frac{U_{0L}^2}{g_x h}, \quad F_y = \frac{U_{0L}^2}{g_y h} \tag{3-6}$$

These parameters represent the ratio of inertia to body force effects.

With primes denoting the (infinitesimal) disturbance quantities, and $U(y)$ being the basic velocity distribution, the velocities, density, and pressure applicable to equations (3-5) are

$$\begin{aligned} u &= U(y) + u'(x, y, t) \\ v &= v'(x, y, t) \\ \rho &= \rho^0(y) + \rho'(x, y, t) \\ p &= p^0(y) + p'(x, y, t) \end{aligned} \tag{3-7}$$

Inserting (3-7) into (3-5) and linearizing, there results from (3-5) the disturbance equations

$$\begin{aligned} \text{a) } \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0 \\ \text{b) } \frac{\partial \rho'}{\partial t} + U \frac{\partial \rho'}{\partial x} + v' \frac{d\rho^0}{dy} &= 0 \\ \text{c) } \rho^0 \left[\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{dU}{dy} \right] &= -\frac{\partial p'}{\partial x} - \frac{\rho'}{F_x} \\ \text{d) } \rho^0 \left[\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} \right] &= -\frac{\partial p'}{\partial x} + \frac{\rho'}{F_y} \end{aligned} \tag{3-8}$$

In arriving at (3-8), equations (2-1) have also been used, in dimensionless form according to the reference quantities (3-4).

Now applying the normal mode approach one may note that since the coefficients in equations (3-8) are independent of x and t these equations admit disturbances of the form

$$q(y) e^{i(kx + \omega t)}$$

or

$$q(y) e^{ik(x + ct)}, \quad c = \omega/k,$$

where $q(y)$ is a complex disturbance amplitude, k the (real) wave number (2π divided by the wavelength of the disturbance), and ω is a complex frequency ($\omega \equiv \omega_r + i \omega_i$). Thus also $c = \omega/k = c_r + i c_i$ is the complex phase velocity of the disturbance. It also follows that ω_i is the amplification rate of the disturbance; with disturbances growing, neutral, or decaying, according to whether ω_i is negative, zero, or positive, respectively. Disturbances of this form may be considered Fourier components of a more general disturbance.

Thus, with the definitions

$$\begin{aligned}
 u' &= f(y) e^{i k(x + ct)} \\
 v' &= h(y) e^{i k(x + ct)} \\
 \rho' &= r(y) e^{i k(x + ct)} \\
 p' &= \pi(y) e^{i k(x + ct)}
 \end{aligned}
 \tag{3-9}$$

the equations (3-8) become, in order, (with $D \equiv \frac{d}{dy}$)

$$\begin{aligned}
 \text{a)} \quad ikf + Dh &= 0 \\
 \text{b)} \quad ik(U + c)r &= -(D\rho^0)h \\
 \text{c)} \quad i\rho^0 k(U + c)f + \rho^0 (DU)h &= -ik\pi - \frac{r}{F_x} \\
 \text{d)} \quad i\rho^0 k(U + c)h &= -D\pi + \frac{r}{F_y}
 \end{aligned}
 \tag{3-10}$$

Equations (3-10) form a system of four ordinary differential equations for the unknown amplitude functions $f(y)$, $h(y)$, $r(y)$, and $\pi(y)$. Since the interest of the analysis is in the behavior of ω (or c) with k and not in the functions themselves, (3-10) is best treated by writing just one differential equation. Since, as will be shown, boundary conditions are described most easily in terms of v' , hence $h(y)$, from (3-10) it is found that $h(y)$ satisfies

$$\begin{aligned} D \left\{ \rho^0 (U + c) Dh - \rho^0 (DU) h - i \frac{(D\rho^0)}{k F_x} \frac{h}{U + c} \right\} - \rho^0 k^2 (U + c) h \\ = - \frac{D\rho^0}{F_y} \frac{h}{U + c} \end{aligned} \quad (3-11)$$

This is the disturbance differential equation for the amplitude function of the y -component of the disturbance velocity. The solution of this equation subject to the boundary conditions, discussed below, will determine a dispersion relation between c and k . For $F_x = \infty$, the results will reduce to those of the usual Kelvin-Helmholtz problem, as will be seen in Chapter IV.

B. Boundary Conditions

The disturbance velocity v' , or $h(y)$, has been chosen as the dependent variable, hence, the differential equation (3-11).

This has been done because physically one is most easily able to interpret the boundary conditions in terms of v' (x, y, t).

Explicitly,

$$y = \eta (x, t) \quad (3-12)$$

describes the equation of the infinitesimally disturbed interface due to the introduction of perturbations of the basic flow. Since the surface described by equation (3-12) is a material surface, i. e., a surface which for all time consists of the same fluid particles, one may write

$$\frac{D}{Dt} (\eta - y) = 0 \quad \text{at } y = \eta$$

or

$$\frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} = \frac{\partial y}{\partial t} \quad \text{at } y = \eta \quad (3-13)$$

Since only initially infinitesimal disturbances are considered, this relation may be evaluated approximately at $y = 0$. Such an approximation is consistent with the previous linearizations of the governing equations since any small quantity may be expanded about $y = 0$, and then linearized. Thus, for example,

$$\left. \frac{\partial \eta}{\partial t} \right|_{y=\eta} = \left. \frac{\partial \eta}{\partial t} \right|_{y=0} + \frac{\partial}{\partial y} \left(\left. \frac{\partial \eta}{\partial t} \right|_{\eta} \right) + \dots$$

Hence,

$$\left. \frac{\partial \eta}{\partial t} \right|_{y=\eta} \approx \left. \frac{\partial \eta}{\partial t} \right|_{y=0}, \quad \text{after linearization.} \quad (3-14)$$

Noting then that

$$\left. \frac{\partial y}{\partial t} \right|_{y=\eta} \approx \left. \frac{\partial y}{\partial t} \right|_{y=0} = v'(x, 0, t),$$

equation (3-13) is written

$$\frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} = v'(x, 0, t) \quad \text{at } y = 0.$$

As with the other disturbance quantities given in equations (3-9), the interface may be described by

$$\eta = s(y) e^{i k(x + ct)} \quad (3-16)$$

Combining (3-15) with (3-16) yields

$$i k(U + c) s(0) = h(0). \quad (3-17a)$$

or

$$s(0) = -\frac{i}{k} \left(\frac{h}{U+c} \right)_{y=0} \quad (3-17b)$$

Since this last equation has been derived without designating the fluid of interest, it obviously is valid for each region of flow. Thus, in order for the displacement of the interface to be uniquely determined, it is required from (3-17b) that

$$\frac{h_u}{U_u + c} = \frac{h_L}{U_L + c} \quad \text{at } y = 0 \quad (3-18)$$

This result is a consequence of the assumption of inviscid fluids. If viscosity were present, it would be required that $U_u(0) = U_L(0)$, and, hence, also $v'_u(0) = v'_L(0)$, as might have been expected.

Equation (3-18) is the first of the required boundary (interface) conditions on v' (or $h(y)$). Others are given by noting that the disturbance must vanish at the wall and as $y \rightarrow \infty$, thus

$$\begin{aligned} \text{a) } h(-d) &= 0 \\ \text{b) } \lim_{y \rightarrow \infty} h(y) &= 0. \end{aligned} \quad (3-19)$$

In addition, a normal stress condition must be satisfied at the disturbed interface. (There is no tangential stress

condition again because of the absence of viscosity.) This normal stress relation is an expression of the force balance between pressure and surface tension forces. In dimensionless form, the condition reads (see [23])

$$-p_u + \frac{1}{W} \frac{\partial^2 \eta}{\partial x^2} = -p_L \quad \text{at } y = \eta, \quad (3-20)$$

where W is the Weber number defined by

$$W = \frac{\rho_{0L} U_{0L}^2 d}{\tau}, \quad (3-21)$$

representing the ratio of inertia to surface tension forces.

In equation (3-20) the pressures given are the total pressures due to the basic flow plus the disturbance flow. Using the superscript (o) to designate basic flow quantities, (3-20) is written

$$-(p_u^o + p_u^i) + \frac{1}{W} \frac{\partial^2 \eta}{\partial x^2} = -(p_L^o + p_L^i) \quad \text{at } y = \eta.$$

As in the derivation of (3-14),

$$p_u'(\eta) \cong p_u'(0)$$

so the above is written

$$-p_u^o(\eta) - p_u'(0) + \frac{1}{W} \frac{\partial^2 \eta}{\partial x^2} = -p_L^o(\eta) - p_L'(0).$$

Using equation (3-15) for η ,

$$-p_u^o(\eta) - p_u'(0) - \frac{k^2}{W} \eta = -p_L^o(\eta) - p_L'(0). \quad (3-21a)$$

From equation (2-7) the basic pressure p^o in each fluid is now determined. In the upper fluid, since the density is a constant

$$p_u^o(\eta) = \text{const.} + \rho_u^o \frac{\eta}{F_y}$$

Assuming an exponential density variation for $y < 0$, i.e.,

$$\rho_L^o = e^{-\beta y},$$

$$p_L^o(\eta) = \text{const.} - \frac{1}{\beta} \frac{\rho_L^o(\eta)}{F_y}$$

However,

$$\rho_L^{\circ}(\eta) = \rho_L^{\circ}(0) + \frac{\partial \rho_L^{\circ}}{\partial y} \Big|_{y=0} \eta = 1 - \beta \eta, \text{ after linearization.}$$

Hence,

$$p_L^{\circ}(\eta) = \text{const.} + \frac{\eta}{F_y}.$$

Using these results for $p_L^{\circ}(\eta)$ and $p_u^{\circ}(\eta)$, equation (3-21a) reads

$$-\frac{\rho_u^{\circ}}{F_y} \eta - p_u^{\prime}(0) - \frac{k^2}{W} \eta = -\frac{\eta}{F_y} - p_L^{\prime}(0). \quad (3-21b)$$

In addition, from (3-10c),

$$-p^{\prime} = \rho^{\circ} (U + c) u^{\prime} - \frac{i}{k} \rho^{\circ} (DU) v^{\prime},$$

and from (3-10a),

$$u^{\prime} = \frac{i}{k} Dv^{\prime}.$$

Thus,

$$-p^{\prime} = \frac{i}{k} [\rho^{\circ} (U + c) Dv^{\prime} - \rho^{\circ} (DU) v^{\prime}].$$

Applying this to equation (3-21b) yields

$$\frac{i}{k} \Delta [\rho^0 (U+c) Dv' - \rho^0 (DU) v'] = + \frac{\Delta(\rho^0)}{F_y} \eta + \frac{k^2}{W} \eta \quad (3-21c)$$

where the following shorthand notation has been used

$$\Delta[\quad] \equiv [\quad]_{y=0^+} - [\quad]_{y=0^-} \quad (3-22)$$

Noting from (3-15) and (3-16) that

$$\eta = - \frac{i}{k} \frac{v'}{U+c}$$

equation (3-21c) may be written finally as

$$\Delta [\rho^0 (U+c) Dv' - \rho^0 (DU) v'] = - \left(\frac{\Delta[\rho]}{F_y} + \frac{k^2}{W} \right) \left(\frac{v'}{U+c} \right)_{y=0} \quad (3-23)$$

The exponential factor, $e^{ik(x+ct)}$, may now be cancelled in the above, giving the final result

$$\Delta [\rho^0 (U + c) Dh - \rho^0 (DU) h] = - \left(\frac{\Delta[\rho^0]}{F_y} + \frac{k^2}{W} \right) \left(\frac{h}{U+c} \right)_{y=0} \quad (3-23a)$$

Thus conditions (3-18), (3-19), and (3-23a) are the required boundary conditions for the solution of the differential

equation (3-11) in each of the two regions of interest. These will be sufficient to determine the dispersion relation, as shown below.

C. The Dispersion Relation

In the discussion of the basic flow, it has been shown that the x-component of velocity may be taken to be an arbitrary function of y. Thus as a first attempt at the solution of (3-11), it is further assumed that the basic velocity profile is uniform in each flow region, but $U_u \neq U_L$ (see Figure 2). This assumption, together with the earlier assumptions discussed in Chapter II, yields from equation (3-11):

(Note: Henceforth the superscript (o) is dropped from the notation.

All densities which appear refer to the undisturbed flow field.)

$$(a) \rho_u (U_u + c) [D^2 h_u - k^2 h_u] = 0 \quad \text{for } y > 0.$$

$$(b) (1+c)(D^2 h_L - k^2 h_L) - \frac{i h_L}{k(1+c)} \left(\frac{D^2 \rho_L}{\rho_L} \right) \frac{1}{F_x} \quad (3-24)$$

$$+ \frac{D \rho_L}{\rho_L} \left[(1+c) D h_L - \frac{i D h_L}{k(1+c)} \frac{1}{F_x} + \frac{h_L}{(1+c) F_y} \right] = 0,$$

for $y < 0$.

Again, as in the derivation of (3-21b),

$$\rho_L = e^{-\beta y}, \quad \text{or} \quad \frac{D\rho_L}{\rho_L} = -\beta. \quad (3-25)$$

Equation (3-24b) may then be written, after some expansion (assuming $c \neq 1$, since c is generally a complex number),

$$\begin{aligned} D^2 h_L + \beta \left(\frac{i}{kF_x(1+c)^2} - 1 \right) Dh_L \\ - \left(k^2 + \frac{i\beta^2}{kF_x(1+c)^2} + \frac{\beta}{F_y(1+c)^2} \right) h_L = 0 \end{aligned} \quad (3-24b')$$

Thus the two differential equations to be solved, viz., (3-24a) and (3-24b'), are ordinary homogeneous second-order constant coefficient equations, yielding the general solutions

$$\begin{aligned} \text{a) } \frac{h_u}{U+c} &= c_1 e^{-ky} + c_2 e^{ky} \\ \text{b) } \frac{h_L}{1+c} &= c_3 e^{p_1 y} + c_4 e^{p_2 y} \end{aligned} \quad (3-26)$$

where p_1 and p_2 are the roots of the quadratic equation

$$p^2 + \beta \left(\frac{i}{kF_x(1+c)^2} - 1 \right) p - \left(k^2 + \frac{i\beta^2}{kF_x(1+c)^2} + \frac{\beta}{F_y(1+c)^2} \right) = 0. \quad (3-27)$$

Hence,

$$p_{1,2} = \frac{\beta}{2} \left(1 - \frac{i}{kF_x(1+c)^2} \right) \pm \frac{1}{2} \left[\beta^2 \left(1 + \frac{i}{kF_x(1+c)^2} \right) + 4 \left(k^2 + \frac{\beta}{F_y(1+c)^2} \right) \right]^{1/2} \quad (3-27a)$$

Inserting the solutions (3-26) into the interface condition (3-23a), and using (3-19a, b) and (3-18), there results

$$\begin{aligned} \rho_u (U_u + c)^2 + \frac{(U_L + c)^2}{k} \left[\frac{p_2 e^{p_2} - p_1 e^{p_1}}{e^{p_2} - e^{p_1}} \right] + i \frac{\beta}{k^2 F_x} \\ = - \frac{(1 - \rho_u)}{kF_y} + \frac{k}{W} \end{aligned} \quad (3-28)$$

Using (3-27a), equation (3-28) may be written finally in the form

$$\begin{aligned}
 & \rho_u (U_u + c)^2 + \frac{(1+c)^2}{k} \left\{ \frac{\beta}{2} \left(1 - \frac{i}{kF_x(1+c)^2} \right) + \frac{1}{2} \left[\beta^2 \left(1 + \frac{i}{kF_x(1+c)^2} \right)^2 \right. \right. \\
 & + 4 \left(k^2 + \frac{\beta}{F_y(1+c)^2} \right)^{1/2} \cdot \coth \frac{1}{2} \left[\beta^2 \left(1 + \frac{i}{kF_x(1+c)^2} \right)^2 \right. \\
 & \left. \left. + 4 \left(k^2 + \frac{\beta}{F_y(1+c)^2} \right) \right]^{1/2} \right\} + \frac{i\beta}{k^2 F_x} = - \frac{(1 - \rho_u)}{kF_y} + \frac{k}{W}
 \end{aligned}
 \tag{3-29}$$

This is the final dispersion relation between the complex phase velocity c and the wave number k . Because of its transcendental nature no general solution can be easily obtained, but limiting cases may be discussed. This is done in the following, together with discussion of the reduction of this equation to the usual Kelvin-Helmholtz results, i.e., those for which the tangential body force is zero ($F_x = \infty$).

In solving equation (3-29) it is noted that for the disturbances of the assumed form, any c with a negative real part implies an unstable motion. That is, if one mode of instability is found, the motion is said to be unstable, regardless of any other modes of the disturbance flow. It is the nature of

these unstable modes and their associated amplification rates
(i.e., $|\omega_i|$) that is sought in the following.

CHAPTER IV

KELVIN-HELMHOLTZ RESULTS FOR INFINITE F_X

Before proceeding with the new results accounting for the tangential component of the body force, the stability results corresponding to the case in which this component is excluded ($F_X = \infty$) will be discussed. The configuration then will be that relevant to the usual Kelvin-Helmholtz flows, that is, two superposed parallel inviscid streams subject to a body force perpendicular to the interface. The motivation for the discussion of this particular situation is two-fold. First, the effect of a lower bounding wall and density stratification of the fluid will be examined. These are two effects which have not been emphasized in the literature relevant to this configuration. (Again, note that the body force component is assumed to be directed from the heavier to the lighter fluid, except where specifically mentioned otherwise.) Secondly, the complete understanding of the stability analysis pertaining to this problem will aid in the interpretation of the results for the complete problem accounting for both body force components. The results for the complete problem, as will be seen, are dependent upon the behavior expected for the reduced ($F_X = \infty$) problem, which is examined in the following.

Thus, with the tangential component of the body force excluded, the characteristic equation (3-29) yields

$$\rho_u (U_u + c)^2 + \frac{(1+c)^2}{k} \left\{ \frac{\beta}{2} + \left[\left(\frac{\beta}{2} \right)^2 + \left(k^2 + \frac{\beta}{F_y (1+c)^2} \right) \right]^{1/2} \right\}$$

$$\coth \left[\left(\frac{\beta}{2} \right)^2 + \left(k^2 + \frac{\beta}{F_y (1+c)^2} \right) \right]^{1/2} \} = \frac{k}{W} - \frac{1 - \rho_u}{k F_y}. \quad (4-1)$$

Note that it is assumed throughout that the density of the upper fluid is less than that of the lower, i.e., $(1 - \rho_u) > 0$.

Usually, however, both fluids are considered to be incompressible and unstratified. In that case $\beta = 0$, and (4-1) yields

$$\rho_u (U_u + c)^2 + (1+c)^2 \coth k = \frac{k}{W} - \frac{1 - \rho_u}{F_y}. \quad (4-2)$$

This is the proper result for the usual Kelvin-Helmholtz problem, with the addition of a bounding wall at $y = -1$. The effect of this wall is seen in the occurrence of the factor $(\coth k)$ in the above expression.

Equation (4-2) may be expressed more conveniently as

$$\begin{aligned} & (\rho_u + \coth k) c^2 + 2(\rho_u U_u + \coth k) c \\ & + \left[\rho_u U_u^2 + \coth k + \frac{1 - \rho_u}{F_y} - \frac{k}{W} \right] = 0 \quad (4-2a) \end{aligned}$$

From equation (4-2a) the results regarding the stability of that configuration are easily determined since now the transcendental nature of equation (4-1) has been eliminated. Since the coefficients of c in (4-2a) are real, if the roots are complex, they will appear as a complex conjugate pair. The root for which $c_i > 0$ corresponds to damped disturbances, whereas $c_i < 0$ corresponds to amplified disturbances. Because the total behavior is a superposition of both solutions, the net effect will be amplification unless $c_i \equiv 0$. Thus the discriminant of (4-2) must be non-negative for c real, yielding the result

$$\frac{k}{W} - \frac{1 - \rho_u}{k F_y} - \frac{\rho_u (U_u - 1)^2}{\rho_u + \coth k} \geq 0. \quad (4-3)$$

This is the necessary and sufficient condition for stability of the configuration applicable to equation (4-2).

Several interesting conclusions may be drawn from this inequality. First of all, if there were initially no flows involved, the configuration would correspond to the "inviscid Rayleigh-Taylor problem" for both fluids unbounded in their extent. Analytically, this corresponds to the result (4-3) less the term

$$\frac{\rho_u^0 (U_u - 1)^2}{\rho_u + \coth k}$$

The criterion for stability then reads (in dimensional form, since now U_R is undefined)

$$k^* \geq k_{\text{cutoff}}^* \quad (4-4)$$

where

$$k_{\text{cutoff}}^* \equiv \sqrt{(\rho_L^* - \rho_u^*) \frac{g_Y}{\tau}}$$

This is the result quoted in the Introduction, showing explicitly the existence of a cutoff wave number due to the stabilizing mechanism of surface tension. Without this effect, $W = \infty$ ($\tau = 0$), and the situation is unstable for all wave numbers.

The same destabilizing behavior of the body force demonstrated by (4-4) for the Rayleigh-Taylor situation also appears for the Kelvin-Helmholtz flow. This behavior is seen explicitly by the term $(1 - \rho_u)/F_y$ in (4-3). This term appears solely as a result of the assumed density discontinuity at the interface. If the direction of the body force were reversed so that the force was oriented toward the heavier fluid, then obviously the Rayleigh-Taylor configuration would always be stable, and the Kelvin-Helmholtz flow would be further (though not completely) stabilized. The effect of inertia, through the term

$$\frac{\rho_u (U_u - 1)^2}{\rho_u + \coth k}$$

represents the classical Kelvin-Helmholtz instability. The appearance of this term may be explained by the fact that there is an assumed jump discontinuity in the velocity at the interface. Its physical significance is best interpreted as a limiting form of the Helmholtz instability, as discussed in Chapter I. It can be seen that the (non-dimensional) velocity difference $(U_u - 1)$ characterizes the strength of vortex sheet which represents the interface, and hence characterizes the Kelvin-Helmholtz instability.

The above conclusions have been emphasized to clarify what it meant when it is said that the flow relevant to the ablation problem is unstable in the Rayleigh-Taylor sense. That is, the orientation of the normal body force is destabilizing in the sense described above for both the Rayleigh-Taylor and the Kelvin-Helmholtz cases. This destabilizing effect is expressed by the term $(1 - \rho_u)/kF_y$ in equation (4-3).

As further illustration, the inequality (4-3) may be written, for the case when the body force is directed toward the denser medium (the liquid), as

$$\frac{k}{W} + \frac{1 - \rho_u}{k F_y} \geq \frac{\rho_u (U_u - 1)^2}{\rho_u + \coth k} \quad (4-5)$$

The left-hand side of this expression is a minimum for (4-5a)

$$k \equiv \tilde{k} = \sqrt{\frac{(1 - \rho_u) W}{F_y}}, \quad (4-5a)$$

so that from (4-5), surface tension will stabilize this Kelvin-Helmholtz flow if

$$\sqrt{\frac{1 - \rho_u}{W F_y}} \geq \frac{\rho_u (U_u - 1)^2}{2(\rho_u + \coth k)} \quad (4-6)$$

This result is due to Kelvin [25] in his study to determine if such a mechanism is the proper one in describing the generation of ocean waves by wind. He concluded that for certain conditions this is indeed a meaningful model. In Kelvin's result, the factor $\coth k$ is replaced by unity because he considered the case for which the lower fluid is unbounded in y , i. e., $d = \infty$. Since $\coth k \geq 1$, from either result (4-3) or (4-6), it can be seen that the presence of the lower wall has a stabilizing effect in comparison with the case without it.

An analogous result to (4-6) cannot be written for the case of interest in the ablation problem in which the body force is directed from the liquid to the gas. This is due to the fact that

$$\left[\frac{k}{W} - \frac{1 - \rho_u}{F_y} \right]$$

has no relative minimum, and hence no value \tilde{k} exists.

In conclusion of the case in which the liquid is unstratified ($\beta = 0$), when the inequality (4-3) is violated, the amplification rate ($\omega_i = c_i k$) of the disturbances, as found from (4-2), is given by

$$\omega_i = c_i k = \frac{k}{\rho_u + \coth k} \sqrt{\rho_u (U_u - 1)^2 + (\rho_u + \coth k) \left[\frac{1 - \rho_u}{k F_y} - \frac{k}{W} \right]} \quad (4-7)$$

Here again the relative stabilizing and destabilizing factors, as discussed, may be ascertained.

Attention may now be turned to the more general result for the stratified medium, $\beta \neq 0$. This result as shown is essentially that found by Alterman [24] under the same conditions as used in deriving (4-1), except that he assumed the upper fluid was also exponentially stratified and that the lower fluid was unbounded vertically. In the present notation, Alterman's result reads

$$\begin{aligned} & \rho_u (U_u + c)^2 \left\{ -\frac{\beta}{2} + \frac{1}{2} \left[\beta^2 + 4 \left(k^2 - \frac{\beta}{F_y (U_u + c)^2} \right) \right]^{1/2} \right\} \\ & + (1 + c)^2 \left\{ \frac{\beta}{2} + \frac{1}{2} \left[\beta^2 + 4 \left(k^2 - \frac{\beta}{F_y (1 + c)^2} \right) \right]^{1/2} \right\} \\ & = \frac{1 - \rho_u}{k F_y} + \frac{k}{W} \end{aligned} \quad (4-8)$$

Note too that he assumes the body force is oriented toward the

heavier medium, that is, in the stabilizing direction. This is seen if the above is compared directly with the present result (4-1).

Alterman, also due to the transcendental nature of his result, does not find an analytical result governing the stability except for the long wavelength case, $k \ll \beta$. Then he determines a criterion on the velocity difference ($U_u - 1$) for stability, which shows that for $k \rightarrow 0$ (infinite wavelength limit) the configuration is always stable.

If the long wavelength limit $k \ll \beta$ is applied in the present case to equation (4-1), the criterion for stability becomes (assuming further that the density ratio ρ_u is very much less than unity)

For $k \ll \beta$, (4-9)

$$a) \frac{k}{W} - \frac{1 - \rho_u}{k F_y} - \frac{\rho_u (U_u - 1)^2}{\rho_u + \coth k} - \frac{\coth \beta/2}{k F_y} \geq 0,$$

or

$$b) 0 < \frac{\rho_u (U_u - 1)^2}{\rho_u + \coth k} \leq \frac{k}{W} - \frac{1 - \rho_u}{k F_y} - \frac{\coth \beta/2}{k F_y}.$$

Again the destabilizing effect of the body force g_Y is seen in the present case through the appearance of the terms

$$\frac{1 - \rho_u}{k F_y} \quad \text{and} \quad \frac{\coth \beta/2}{k F_y}$$

The form (4-9b) of the result is analogous to Alterman's in the sense that it indicates the allowable relative velocities $|U_u - 1|$ for a stable configuration. The effect of the terms involving F_y is to restrict this range of allowable relative velocities, thereby implying a destabilizing influence of the body force g_y . Further, because of this destabilizing effect the result (4-9) indicates complete instability in the long wavelength limit, $k \rightarrow 0$. It is also apparent from this latter result that stratification enhances the instability through the inclusion of the term $\frac{\coth \beta/2}{k F_y}$, which serves as an additive effect to the usual Rayleigh-Taylor mechanism expressed by the

$$\frac{1 - \rho_u}{k F_y}$$

Therefore, density stratification acts to further enhance the Rayleigh-Taylor destabilizing effect on the present Kelvin-Helmholtz situation. Conversely, if the body force were directed toward the heavier fluid, then both terms involving F_y would become stabilizing factors, and complete stability would result in the limit $k \rightarrow 0$, as indicated by Alterman.

In addition, it is noted that the other extreme limit for short wavelengths ($k \gg \beta$) reduces the general result (4-1) to the criterion for stability given by

$$\frac{k}{W} - \frac{1 - \rho_u}{k F_y} - \frac{\rho_u (U_u - 1)^2}{\rho_u + \coth k} - \frac{\beta}{2k^2 F_y} \geq 0, \text{ for } k \gg \beta. \quad (4-10)$$

Here, because of the limit taken, the additional term $\frac{\beta}{2k^2 F_y}$ may be considered negligible compared with the other terms of the inequality. Its appearance in (4-10) does, however, indicate a destabilizing influence of stratification, as in the previous limit, $k \ll \beta$. The result (4-10) may thus be considered essentially the same as (4-3), derived for no density stratification ($\beta = 0$). Note that complete stability is predicted in the short wavelength limit $k \rightarrow \infty$

Therefore, it may be concluded from the comparisons made of the criteria (4-9) and (4-10) that small wavelength disturbances are more stable than those for large wavelengths. One contributing factor to this result, not yet emphasized, is that for the small wavelength case ($k \gg \beta$) surface tension has a more dominant role. This fact is seen explicitly in the term k/W

appearing in both limiting results. Finally, the density stratification appears as a destabilizing mechanism in every case, with its effect most dominant in the long wavelength limit ($k \ll \beta$). Again, this destabilization is due to the assumed orientation of the body force g_Y (viz., toward the lighter medium), which causes the stratified layer itself to be unstable.

In conclusion, note that in all of the results the Kelvin-Helmholtz destabilizing influence appears through the terms involving the relative velocity ($U_u - 1$). That this Kelvin-Helmholtz mechanism appears independently of the Rayleigh-Taylor mechanism is obvious from any of the results (4-3, 9, 10). It was this fact that led to the statement in the Introduction that the effect of the normal component of the body force (through the terms involving F_y) should be considered as a superposition of the Rayleigh-Taylor mechanism on the Kelvin-Helmholtz problem. In the next chapter it will likewise be shown that the influence of the tangential component of the body force also appears as a superposition on the Kelvin-Helmholtz instability described in the present chapter.

CHAPTER V

KELVIN-HELMHOLTZ RESULTS FOR FINITE F_x

A. Introduction

In Chapter IV the pertinent stability results for the case of zero tangential body force ($F_x = \infty$) were discussed. It was seen there how, under properly simplified circumstances, the dispersion relation could be reduced to forms which yield general stability criteria (e.g., (4-3)) and/or amplification rates (e.g., equation (4-7)). With attention now focused on the general dispersion relation (3-29), it is easily seen that no such simple results are obtained in general. The cause of this dilemma lies, of course, in the transcendental nature of that equation. The fact that the "characteristic" parameter c appears within transcendental functions makes its determination in a closed analytical form impossible. Moreover, any trends indicating the dependence of c on the relevant parameters (e.g., F_x , F_y , and W) are equally difficult to obtain.

In handling (3-29), however, two alternatives are possible. First, since this is an equation involving the complex constant $c = c_r + i c_i$, one could separate it into two real equations; and then, by numerical means, calculate the amplification rates

(c_i, k) corresponding to various sets of non-dimensional parameters. In this way some idea of the behavior of the system can be ascertained for at least some situations of interest. In the present case the appearance of the hyperbolic cotangent makes such a procedure quite laborious. The numerical work requires an iteration scheme simultaneously on the two real equations, which both still contain transcendental functions of c_i and c_r . The rather uncertain values that should be chosen for several of the parameters (especially U_u and W) make such numerical work of limited value.

The second alternative is to reduce (3-29), by proper limiting arguments, to a form analogous to those obtained in Chapter IV, but now including all pertinent factors. It will be seen that such results represent the solutions indicated in Chapter IV with additional terms for finite F_x included. This will be in the same sense that (4-9) and (4-10) were generalizations of (4-3) for $\beta \neq 0$. The results to be obtained in the following appear valid for the ablation problem taken in a proper limiting form.

B. Simplification of the General Result (3-29)

In order to reduce systematically the dispersion relation (3-29) to a workable form relevant to the ablation problem,

representative data must be introduced. These are given in the following (see ref. [2]),

Flight Mach number: 18

Altitude: 90,000 feet

Sound speed: 987 feet/sec.

Gas density (ρ_u^*): 5.44×10^{-5} slugs/feet³

Deceleration force (g): 23 G's = 735 feet/sec.²

Gas speed (U_u): 3,000 feet/sec.

(5-1)

Liquid density (ρ_{0L}^*): 4.07 slugs/feet³

Liquid film speed (U_L): 2.0 feet/sec.

Liquid film thickness (d): 0.1 feet.

Surface tension coefficient (τ): 0.021 lb./feet

Exponential density factor (β^*): 30/feet.

For these values the dimensionless parameters, as defined earlier, become for an angle $\theta = 35^\circ$ on the body as in Figure 1,

$$F_x = 0.067$$

$$F_y = 0.092$$

$$W = 77.5$$

$$\rho_u = 1.34 \times 10^{-5}$$

$$U_L = 1$$

$$U_u = 1500$$

$$\beta = 3$$

(5-2)

The values now given in (5-2) readily allow some simplification of (3-29). First, the extremely large value of U_u dominates the phase velocity c , so that the first term of (3-29) may be written

$$\rho_u (U_u + c)^2 \approx \rho_u U_u^2 . \quad (5-3a)$$

The ensuing results will show that c is a quantity of unit order, thereby further justifying this approximation. From the governing disturbance differential equation (3-8d), applied to the gas stream, the above approximation is equivalent to the statement

$$\frac{\partial v'_u}{\partial t} \ll U_u \frac{\partial v'_u}{\partial x} , \quad (5-3b)$$

namely, that the unsteady part of the inertia may be neglected compared with the convective term dominated by the influence of the large uniform gas velocity (ratio) U_u . This assumption appears quite valid in the present context. Chang and Russell [7] also claim its validity if $\rho_u \ll 1$, which is certainly the case here.

As a consequence of (5-3), the relation (3-29) may now be written

$$\begin{aligned}
 & (1+c)^2 \left\{ \frac{\beta}{2} \left(1 - \frac{i}{kF_x(1+c)^2} \right) + \frac{1}{2} \left[\beta^2 \left(1 + \frac{i}{kF_x(1+c)^2} \right)^2 \right. \right. \\
 & \quad \left. \left. + 4 \left(k^2 + \frac{\beta}{F_y(1+c)^2} \right) \right]^{1/2} \cdot \coth \frac{1}{2} \left[\beta^2 \left(1 + \frac{i}{kF_x(1+c)^2} \right)^2 \right. \right. \\
 & \quad \left. \left. + 4 \left(k^2 + \frac{\beta}{F_y(1+c)^2} \right) \right]^{1/2} \right\} = \frac{k^2}{W} - \frac{i\beta}{kF_x} - \frac{1-\rho_u}{F_y} - k\rho_u U_u^2
 \end{aligned}$$

(5-4)

This result is still not in an easily workable form for analysis. For further simplification, use may be made of the assumption (stated in Chapter II) which allowed the x-component of the pressure gradient to be neglected in a local region of interest. If this assumption is to be valid within the framework of the stability analysis, the wavelengths of the allowed disturbances must be "short enough" that they are unaffected by the x-component of the pressure gradient. More precisely, the length characteristic of a pressure change, given by

$$\left(\frac{1}{p} \left| \frac{\partial p}{\partial x} \right| \right)^{-1} ,$$

is assumed to be very large compared with a disturbance wavelength λ . Thus the criterion for validity of the above assumption is that

$$\frac{\lambda}{p} \left| \frac{\partial p}{\partial x} \right| \ll 1. \quad (5-5)$$

Therefore, one may infer that the stability analysis is best considered in the small wavelength (large k) limit.

If (5-5) is now applied to the lower fluid, $|\partial p / \partial x|$ may be replaced by ρ_L / F_x and p by an "effective dynamic pressure" $\rho_L |1+c|^2$. The inequality (5-5) may then be written

$$\frac{1}{F_x} \ll \frac{|1+c|^2}{\lambda} \equiv \frac{k}{2\pi} |1+c|^2 < k |1+c|^2 \quad (5-5a)$$

Further explanation of this last inequality can be found by examining the disturbance differential equation (3-24b) for the liquid layer. One then sees that (5-5) implies the term

$$\frac{i \beta Dh_L}{k F_x (1+c)^2}$$

is neglected in comparison with βDh_L . Thus (5-5a) is tantamount

to neglecting the effect of heterogeneity of the fluid on the body force influence as compared with the inertial influence. If this latter argument is also applied to the other body force component, it may be further assumed that

$$\frac{\beta}{F_y} \ll k^2 |1 + c|^2. \quad (5-6)$$

Note, however, that even with the assumptions (5-5a) and (5-6), the right-hand side of equation (5-4) contains both body force components, where both these terms arise as a result of the assumed density discontinuity at the interface. The appearance of these terms is independent of the latter assumptions.

C. Results

With the assumptions (5-5) and (5-6) applied to the dispersion relation (5-4) there results

$$(1+c)^2 = \frac{1}{F(k, \beta)} \left\{ \frac{k^2}{W} - \frac{i\beta}{kF_x} - \frac{1-\rho_u}{F_y} - k \rho_u U_u^2 \right\} \quad (5-7)$$

where

$$F(k, \beta) \equiv \frac{\beta}{2} + \left[k^2 + \frac{\beta^2}{4} \right]^{1/2} \coth \left[k^2 + \frac{\beta^2}{4} \right]^{1/2}.$$

The right-hand side of (5-7) is a complex quantity, independent of the phase velocity c , so that setting $c = c_r + i c_i$ allows the separation of this equation into two real equations.

From these equations it is found directly that

$$(a) \quad c_i^2 = \frac{k}{2F(k, \beta)} \left[\left(\rho_u U_u^2 + \frac{1-\rho_u}{kF_y} - \frac{k}{W} \right) + \sqrt{\left(\rho_u U_u^2 + \frac{1-\rho_u}{kF_y} - \frac{k}{W} \right)^2 + \left(\frac{\beta}{k^2 F_x} \right)^2} \right] \quad (5-8)$$

$$(b) \quad c_r = -1 - \frac{1}{F(k, \beta) c_i} \left(\frac{\beta}{2kF_x} \right)$$

From this last result it may be noted that $c_r = -1 + O(1/k^2)$ or $c_r^* \approx -U_L$, for large k (small wavelengths). From the assumed form of the variation of the disturbance this result yields

$$q(y)e^{ik(x+ct)} = q(y)e^{-c_i t} \cdot e^{ik(x+c_r t)} \approx q(y)e^{-c_i t} e^{ik(x-t)},$$

so that the disturbances may be considered waves of amplitude $q(y)e^{-c_i t}$ traveling in the positive x -direction with the velocity of the liquid layer. Hence, relative to this layer, the disturbances are essentially stationary. This result may serve as justification for those analyses which treat the liquid as an initially stationary

medium and study disturbances relative to it. The work of Chang and Russell [9] is an example of such.

In order to interpret the result (5-9a) for c_i^2 further, it may be noted that the stability criterion (4-10), relevant to the Kelvin-Helmholtz problem with zero tangential body force component ($F_x = \infty$), may be written for the present case (with $U_u \gg 1$, $\rho_u \ll 1$, and large k) as

$$\rho_u U_u^2 + \frac{1 - \rho_u}{k F_y} - \frac{k}{W} \leq 0. \quad (5-9a)$$

The above expression also appears as a factor in the expression (5-8a) for c_i^2 . From the numerical data (5-2) the above may be written

$$30.2 + \frac{10.9}{k} - \frac{k}{77.5} \leq 0. \quad (5-9b)$$

From either of these forms a cutoff wave number, analogous to that derived in Chapter IV, can be found. If this is done, there results from (5-9a, b)

$$k \geq \bar{k} \text{ for stability,} \quad (5-9c)$$

where

$$\bar{k} \equiv \frac{\rho_u U_u^2 W}{2} + \sqrt{\left(\frac{\rho_u U_u^2 W}{2}\right)^2 + \frac{(1 - \rho_u)W}{F_y}} = 2340.$$

For $k < \bar{k}$, that is, the condition for unstable modes in the $F_x = \infty$ case, the first factor in c_i^2 as given in (5-9a) is positive thereby adding to instability of the complete configuration ($F_x < \infty$). Thus, for $k < \bar{k}$, there is an "inherent instability" which acts independently of the existence of the tangential body force. This "inherent instability", as seen from (5-9) is due primarily to the large velocity discontinuity at the interface. Thus, except for very small wavelengths (viz., $\lambda \leq \bar{\lambda} = 2\pi/\bar{k}$), all disturbances grow, regardless of the stabilizing influence of surface tension. This latter mechanism, of course, does provide a means of lowering the amplification rates, and most importantly, admitting the cutoff wave number \bar{k} .

In addition, (5-10) indicates that the tangential component of the body force adds to the "inherent instability". This destabilizing influence appears to be independent of the assumed directional sense of this component.

For $k < \bar{k}$, since $\rho_u U_u^2 \gg \beta/k^2 F_x^2$, equation (5-8a) may be further approximated by

$$c_i^2 = \frac{k}{F(k, \beta)} \left[\left(\rho_u U_u^2 + \frac{1-\rho_u}{kF_y} - \frac{k}{W} \right) + \frac{(\beta/2k^2 F_x^2)^2}{\rho_u U_u^2 + \frac{1-\rho_u}{kF_y} - \frac{k}{W}} \right], \quad (5-10a)$$

hence,

$$\omega_i^2 = c_i^2 k^2 = \frac{k}{F(k, \beta)} \left[k^2 \left(\rho_u U_u^2 + \frac{1-\rho_u}{kF_y} - \frac{k}{W} \right) + \frac{(\beta/2kF_x)^2}{\rho_u U_u^2 + \frac{1-\rho_u}{kF_y} - \frac{k}{W}} \right]. \quad (5-10b)$$

Similarly, for $k > \bar{k}$,

$$\omega_i^2 = \frac{1}{kF(k, \beta)} \left[\frac{(\beta/2F_x)^2}{\frac{k}{W} - \rho_u U_u^2 - \frac{1-\rho_u}{kF_y}} \right] \quad (5-11)$$

so that the amplification rate in this case goes to zero as $k \rightarrow \infty$. (Note that as $k \rightarrow \infty$, $F(k, \beta)$ tends to k .) This is the expected behavior, namely stability for the very short wavelength disturbances due to surface tension effects.

Since the results of this analysis are generally valid for the small wavelength limit (large k), the result (5-11) is a proper one. Also, because of the rather large value indicated for \bar{k} , the result (5-10) will also remain valid over some wide range of values for k , i.e., so long as the inequalities (5-5) are not violated.

Also, from the definition of $F(k, \beta)$ in (5-7a), the ratio $[k/F(k, \beta)]$ appearing throughout the results is less than unity for any finite wave number k . This serves to lower the amplification rates, as seen in equations (5-10) and (5-11). This stabilizing tendency is again primarily due to the finite thickness of the liquid layer.

If the amplification rate given by (4-7) is reduced by the present data, as in deriving (5-10), there results

$$\omega_i^2 \Big|_{F_x = \infty} = k^2 (\rho_u U_u^2 + \frac{1-\rho_u}{kF_y} - \frac{k}{W}),$$

so that (5-10b) may be written

$$\omega_i^2 = \frac{k}{F(k, \beta)} \left[\omega_i^2 \Big|_{F_x = \infty} + \frac{(\beta/2 F_x)^2}{\omega_i^2 \Big|_{F_x = \infty}} \right] \text{ for } k < \bar{k}. \quad (5-12)$$

This result is, of course, valid only for long waves such that $k < \bar{k}$, i.e., waves long enough so that the "inherent instability" indicated by (5-10) plays a role, but not so long that either of the inequalities (5-5) or (5-6) are violated. For the shorter waves ($k \geq \bar{k}$), the amplification rate for the case of zero tangential body

force ($F_x = \infty$) is zero, and the amplification rate for the complete problem ($F_x < \infty$) is given by (5-11).

D. Summary and Comparisons

In order to reduce the general dispersion relation (3-29) to a workable form, two important assumptions were made. First, the transient motion of the gas was neglected in comparison with its inertial effect. Secondly, the inequalities (5-5 a, b) were applied, having made use of the previous assumption that the pressure gradient in this flow direction could be neglected locally.

Two results followed, depending on the satisfaction of the inequality (5-9), which expresses the stability of the present configuration with zero tangential body force ($F_x = \infty$). For the violation of this inequality ($k < \bar{k}$), it was found that the situation is "inherently unstable", i.e., unstable independently of the tangential component of the body force. This component was then found to be further destabilizing for the $k < \bar{k}$ case. For the inequality (5-9) satisfied ($k \geq \bar{k}$) there is no "inherent instability", but the tangential body force was found again to be destabilizing. Even in this case, however, from (5-11), the relative influences of the other parameters are seen. The destabilizing effect of the

tangential body force component appeared to be more dominant for the situation which is "inherently unstable". It should be noted here that the effect of this "inherent instability" is not unlike the superposition of the Rayleigh-Taylor mechanism on the Kelvin-Helmholtz instability, as discussed previously. Therefore, the "inherent instability" can be interpreted as just the effect of the Kelvin-Helmholtz instability discussed in Chapter IV superposed onto the more general case for a non-zero tangential body force component.

The results presented herein are indicative of the behavior expected for the configuration corresponding to the ablation problem, i. e., subject to the representative data given by (5-2). The amplification rates as given by equations (5-10) and (5-11), for the conditions specified, indicate the behavior of the growth of the disturbances as a function of the wave number. These results also indicate the dependence of the growth rates on the dimensionless parameters of importance, viz., F_x , F_y , W , ρ_u , and U_u . The rates given by (5-10) and (5-11) are asymptotic (i. e., for the short wavelength limit) in nature, due to the aforementioned assumptions and limiting cases. The trends predicted appear valid when comparison is made with

Cheng [2] , who accounted for the tangential component of the body force, with viscous effects included. This comparison can, of course, be made only for the short wavelength limit where, however, the viscous effects neglected in the present analysis serve as a stabilizing influence (cf. Miles [16]).

Cheng [2] has found, using an approximate method of analysis derived from consideration of the disturbance energy transfer, that whether the tangential body force field is destabilizing or stabilizing depends only on whether the product

$$\left(\frac{-1}{F_x} \right) \left(\frac{d^2 U_0 / dy^2}{d U_0 / dy} \right)_{y=0^+}$$

is a positive or negative. ($U_0(y)$ is the interfacial mass velocity, which is continuous because the velocity across the viscous interface will suffer no jump discontinuity.) In the present case, F_x has been chosen as a positive quantity, and hence for stability it is required that

$$\left(\frac{d^2 U_0 / dy^2}{d U_0 / dy} \right)_{y = 0^+} > 0.$$

Cheng, however, has found that this ratio is negative at the stagnation point, and appears to decrease in the downstream

direction, so long as one stays in the subsonic nose region of a blunt body, see Fig. 1. Thus, Cheng concludes that the tangential component is always destabilizing in this region. Therefore, the results of the present analysis do indicate the proper behavior of this body force component in the nose region of a blunt body.

Further, Cheng points out that any propagating disturbances ($c_r^* \neq U_0$) of the Helmholtz type can always be suppressed by sufficiently increasing the viscosities of the media, regardless of destabilizing effect of the tangential gravity field. In the present analysis, the "Helmholtz instabilities" are expressed in terms of the Kelvin-Helmholtz mechanism characterized by the relative velocity ($U_u - 1$). Thus, in the absence of viscosity, this velocity difference is fixed, and serves only as destabilizing influence.

Finally, Cheng has indicated that stationary disturbances ($c_r^* = U_0$) of a Rayleigh-Taylor type may be excited by both the normal and tangential components of the body force. These disturbances, however, cannot be eliminated completely, no matter how large the viscosity. This is the expected behavior for an unbounded Rayleigh-Taylor situation. Only the introduction of a cutoff wave number, above which all disturbances are damped,

can stabilize such a situation. Cheng thus concludes that it is these latter stationary disturbances which are of the most concern in the flow instabilities relevant to the problem of melting ablation. In the present analysis, no distinction between the two types of disturbances considered by Cheng can be made. The reason is that Cheng's reference velocity is a uniquely defined interface velocity which is not available herein. Recall, however, that for very small wavelengths the disturbances were shown to be essentially stationary with respect to the fluid layer. Unfortunately, this latter result cannot be used as a basis for comparison with Cheng's conclusions.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The interfacial stability problem of two parallel superposed inviscid fluid streams has been studied with the purpose of determining the effect of a body force component tangential to the flow direction. In seeking this effect, other stabilizing and destabilizing influences, such as surface tension, density stratification, a relative interface velocity, a lower bounding wall, and a perpendicular body force component, have been analyzed.

The linearized normal mode theory was used for the stability analysis. This theory yielded a boundary-value problem consisting of two second-order ordinary differential equations subject to homogeneous boundary conditions and to a single interface condition. In arriving at these results, it was necessary to assume that the pressure gradient in the flow direction could be neglected in local regions of interest. This assumption was later used to show that the wavelengths of the allowed disturbances must be small compared with the length characteristic of a pressure difference in the flow direction. The result following from solution of the boundary-value problem was a dispersion relation (3-29) between the complex phase velocity and the wave number of the

disturbances. Several dimensionless parameters, characterizing the effects of the body force components, surface tension, the relative interface velocity, and the density ratio, appear within the dispersion relation. These parameters are later used to determine the relative stabilizing or destabilizing influence of the associated physical mechanisms.

The transcendental nature of the dispersion relation (3-29) made a completely general analytical result impossible. After making several well-justified simplifications, the influence of the important parameters was distinguished. By studying first the case with zero tangential component of the body force, the perpendicular component was shown to be stabilizing or destabilizing depending on whether it was directed toward or away from the heavier fluid. Also, when this perpendicular component is directed away from the heavier fluid, the effect of density stratification of that fluid is to enhance the already present instability (due to the discontinuity in density at the interface). Conversely, if the direction of this force is reversed, stratification acts as a stabilizing influence. The effects of a bounding wall on the lower fluid and of surface tension at the interface were found to be always stabilizing. Most importantly, the velocity discontinuity

at the interface implied the always destabilizing Kelvin-Helmholtz mechanism which is characterized by the magnitude of the relative interface velocity. Amplification rates for the several unstable situations were determined as functions of the dimensionless parameters associated with the above-named effects.

In studying the more general case including a non-zero tangential component of the body force, it was shown that this component is always destabilizing. This result was found to be independent of the assumed directional sense of the tangential component. Also, a cutoff wave number was shown to exist, above which disturbances are destabilized only because of the presence of the tangential body force. The effects of surface tension, the perpendicular body force, and the relative interface velocity do, however, play a role in determining the amplification rates associated with such disturbances. For those disturbances below the cutoff wave number, as "inherent instability" occurs as a result of the Kelvin-Helmholtz mechanism described for the case of zero tangential body force component. The tangential component was then found to act as a further destabilizing mechanism. Again, amplification rates for these relevant cases were determined.

Hence, the present work has been able to show some significant features affecting the stability of the generalized Kelvin-Helmholtz model assumed. Relative stabilizing and destabilizing influences have been delineated, and a certain understanding of the importance of the several physical mechanisms involved was achieved.

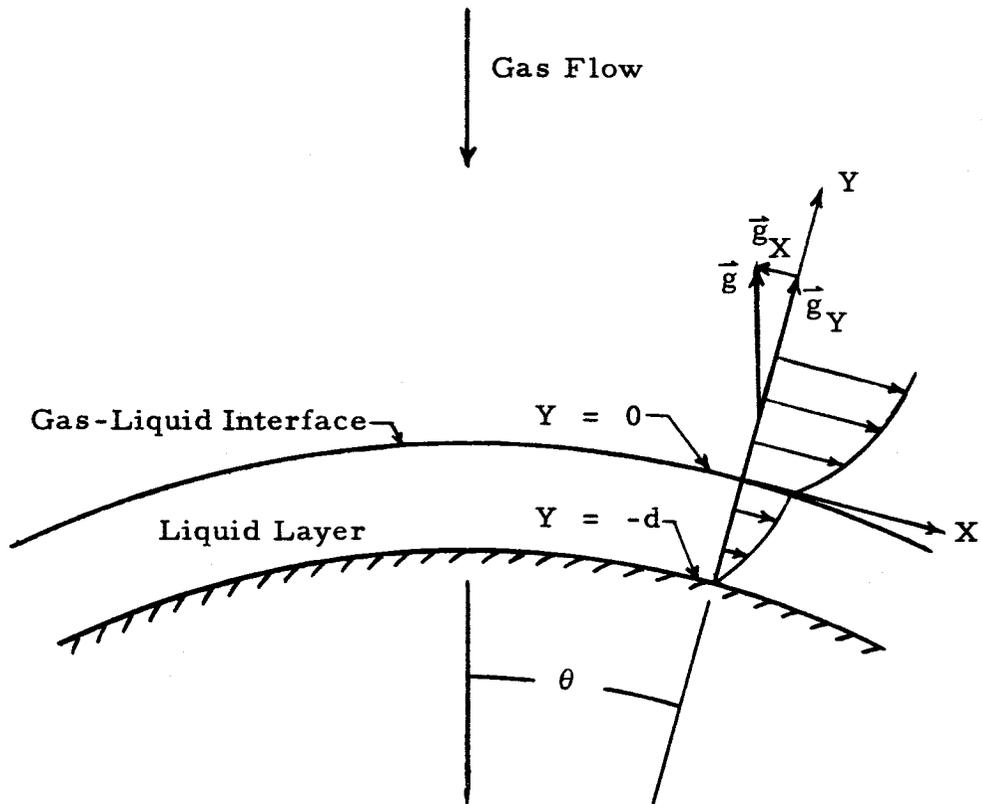


Figure 1

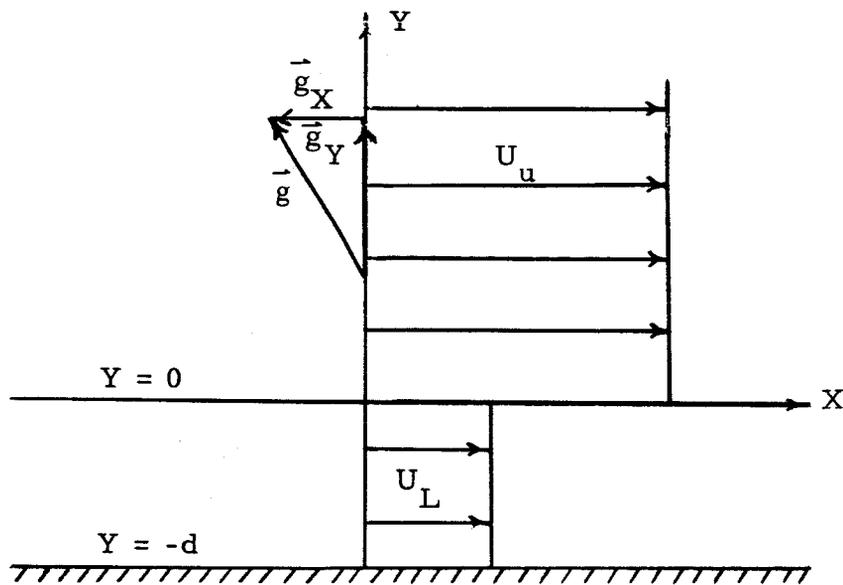


Figure 2

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